Physics 403 Problem Set 1

Model Solution

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Problem 1. Discrete Probabilities

(a) *Suppose you throw eight equally weighted six-sided dice. What is the probability that I will get "1" showing up twice, "3" showing up once, "4" showing up twice, "5" showing up twice, and "6" showing up once?*

Solution: We shall denote the outcome of a roll using the list $r = (r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8)$, where r_i is the value rolled by the i^{th} die. Notice that if two entries r_i and r_j in the list are equal, then exchanging them in the list leaves r unchanged. This tells us that the number of distinct permuatations is given by the multinomial coefficient $C^8_{n_1 \dots n_6}$ where n_i is the number of times that i is rolled. The number of distinct permutations of the roll $r = (1, 1, 3, 4, 4, 5, 5, 6)$ is then:

$$C^8_{201221} = \frac{8!}{2!0!1!2!2!1!} = 5040 \tag{1.1}$$

Thus, there are 5040 distinct ways to roll the combination of numbers. As each roll configuration is equally probable, the probability of rolling a particular combination is the number of ways to roll that combination divided by the total number of rolls. Each die can have 6 outcomes, and the value on each die is independent of the others. The total number of outcomes is subsequently $6^8 = 1\,679\,616$.

We conclude that the probability P_r of rolling the listed combination of numbers is 5040/1679616, which we can write as the percentage:

$$\boxed{P_r = 0.3\%} \tag{1.2}$$

(b) *Suppose you are dealt a hand of five cards from a deck of cards. What is the probability that you will get exactly three cards of the same value?*

First, we shall determine the number of threes of a kind. For a given value, a three of a kind must include three suits of that value and exclude one. We have four choices of suits and must choose three, and the number of ways to do this is $\binom{4}{3} = 4$. There are 13 values with which we can make threes of a kind, so there are 52 total threes of a kind.

Next, we consider the number of ways we could draw the other two cards. Out of the 49 remaining cards in the deck, one is the same as the three of a kind drawn. Then, we must draw 2 out of 48 cards. The number of ways we can do this is given by the binomial coefficient $\binom{48}{2} = 1128$. Combined with our previous result, we see that there are $52 \times 1128 = 58656$ hands with three of a kind.

The probability of drawing three of a kind is equal to the number of ways to draw three of a kind divided by the total number of possible hands. We are drawing 5 cards out of 52 possible cards, which means there are $\binom{52}{5} = 2598960$ hands we can draw. Then, the probability P_3 of drawing three cards of the same value is $58656/2598960$, which, as a percentage, is:

$$\boxed{P_3 = 2.257\%} \tag{1.3}$$

(c) *Suppose you get exactly three of a kind on the first draw, and you discard the two other cards. You then draw two more cards to replace these. What is the probability that these two other cards will be a pair? What is the probability that one of these two other cards will have the same value as the three of a kind you already have?*

There are two cases we must consider. First, we consider the case that the two discarded cards were a pair. We shall now count the number of ways we can draw a pair. There is one pair of cards with value of the discarded pair, and no pairs with value of the three of a kind. There are 11 other values for which all four suits remain, and for each of these values, there are $\binom{4}{2} = 6$ pairs. Thus, we have $6 \times 11 + 1 = 67$ pairs this way. As there are $\binom{47}{2} = 1081$ ways we can draw two cards, the probability C_1 of drawing a pair is $67/1081 = 0.062$.

Now we consider the case that the discarded cards were not a pair. In this case, there are 10 values with all four suits remaining. The number of pairs of these cards that can be drawn is $10 \times \binom{4}{2} = 60$. There are 2 values for which three suits remain. The number of pairs of these cards that can be drawn is $2 \times \binom{3}{2} = 6$. The remaining suit cannot have any pairs of it drawn. Once more, the way we can draw two cards is 1081, so the probability C_2 of drawing a pair is $66/1081 = 0.0611$.

Now, we must consider the probability of each of these scenarios happening. Let P_1 be the probability of drawing three of a kind with the other two cards being a pair (the first case) and P_2 be the probability of drawing three of a kind with the other two cards not being a pair (the second case). Then, as we know that we start with a three of a kind, the probability of starting in case one is $P_1/(P_1 + P_2)$, and the probability that we started in case 1 and drew a pair is $C_1 P_1/(P_1 + P_2)$. Similarly, the probability of starting in case two is $P_2/(P_1 + P_2)$ and the probability of case two happening is $C_2 P_2/(P_1 + P_2)$. Then, the total probability P_{32} of starting with a three of a kind, discarding two cards, and drawing a pair is:

$$P_{32} = \frac{C_1 P_1}{P_1 + P_2} + \frac{C_2 P_2}{P_1 + P_2} \tag{1.4}$$

We must now determine P_1 and P_2. We shall start with P_1. If three of the cards have the same value, then the other pair could be one of 12 values. Each of these values has four suits, so there are $12 \times \binom{4}{2} = 72$ ways that these cards could be a pair. Then, recalling from the previous part that there are 52 threes of a kind, the probability of drawing five cards, having three be of one value and two be of another, is $72 \times 52/\binom{52}{5} = 0.00144$.

Next, we shall determine P_2. If three of the cards have the same value, then the two remaining cards must be two distinct values out of the remaining 12. There are $\binom{12}{2} = 66$ ways that these can be drawn. There are 4 suits for each of these values, so there are $66 \times 4 \times 4 = 1056$ ways to draw these pairs. There are 52 ways to draw a three of a kind, so the probability of drawing three of a kind without a pair is $52 \times 1056/\binom{52}{5} = 0.0211$. Substituting this into our expression for P_{32}, we find:

$$P_{32} = \frac{0.00144 \times 0.062}{0.0211 + 0.00144} + \frac{0.0211 \times 0.0611}{0.0211 + 0.00144} \tag{1.5}$$

We can now evaluate the right hand side to the following percentage:

$$\boxed{P_{32} = 6.12\%} \tag{1.6}$$

We shall now determine the probability of drawing a card with the same value as the three of a kind. There is one card left in the deck that satisfies this criterion, so one of the cards must be this card. There are 46 other cards we can draw, which leaves us with 46 possible draws to obtain a four of a kind. There are $\binom{47}{2} = 1081$ total pairs we can draw, so the probability P_4 of obtaining the fourth card is 46/1081, which, in percentage form, we can write as:

$$\boxed{P_4 = 4.26\%} \tag{1.7}$$

Problem 2. Thermodynamics for a Magnetic System

Suppose we have a magnetic system with the equation of state:

$$M(T, B) = \frac{CB}{T} \tag{2.1}$$

where M is the magnetization, B is the magnetic field, T is a temperature, and C is some constant. The energy U of this system is:

$$U = -MB \tag{2.2}$$

If the field B is changed by an amount dB, the work dW done by the system is given by:

$$dW = M\,dB \tag{2.3}$$

(a) *Show that the heat given to the system under simultaneous changes dB and dM satisfies $dQ = -B\,dM$.*

The first law of thermodynamics tells us that:

$$dU = dQ - dW \tag{2.4}$$

Writing dW in terms of the magnetization and the magnetic field, we find:

$$dU = dQ - M\,dB \tag{2.5}$$

Next, using the product rule, we can differentiate our expression for the internal energy U to find that:

$$dU = -M\,dB - B\,dM \tag{2.6}$$

Substituting this into the left hand side of our previous expression and solving for dQ, yields:

$$\boxed{dQ = -B\,dM} \tag{2.7}$$

(b) *Using the previous result, find the differential change in the entropy dS and the form $S(M)$ for the entropy of the system.*

The differential dS of the entropy is given by:

$$dS = \frac{dQ}{T} \tag{2.8}$$

Substituting in our expression from the previous part, we find:

$$\boxed{dS = -\frac{B}{T}dM} \tag{2.9}$$

Now, we can rearrange the equation of state for the system to find:

$$B = \frac{MT}{C} \tag{2.10}$$

Substituting this into our expression for the entropy, we have:

$$dS = -\frac{M}{C}dM \tag{2.11}$$

Finally, integrating both sides, obtain the following expression for the entropy:

$$\boxed{S(M) = S(0) - \frac{M^2}{2C}} \tag{2.12}$$

Problem 3. N Spin-1/2 Systems

Consider a set of N non-interacting spin-1/2 systems in a magnetic field, such that the energies of each of the individual spins are E_1 and E_2 respectively.

(a) *Find the partition function for this system. Find the average energy U of the total system at a temperature T. From this expression, derive the specific heat C_V.*

We recognize that the particle number is being held constant, so we considering the canocical ensemble. Then, the partition function Z is given by:

$$Z = \sum_{\text{states}} e^{-\beta E_s} \tag{2.1}$$

where E_s is th energy of the state. For a state s, let s_1 and s_2 denote the number of spin systems in the first state and second state respectively. Then, the energy E_s of the state is:

$$E_s = s_1 E_1 + s_2 E_2 \tag{2.2}$$

Notice that the spin systems must either be in the s_1 or the s_2 spin state, so we have the constraint that:

$$N = s_1 + s_2 \tag{2.3}$$

Then, our energy E_s can be written in the following way:

$$E_s = s_1 E_1 + (N - s_1) E_2 \tag{2.4}$$

Substituting this into our expression for the partition function, we find:

$$Z = \sum_{\text{states}} e^{-\beta s_1 E_1} e^{-\beta (N - s_1) E_2} \tag{2.5}$$

We have written the energy of the state such that it only depends on s_1, so we can express the sum over states as a sum over the allowed values of s_1. The minimum allowed value of s_1 is zero, and the maximum allowed value is N. Then:

$$Z = \sum_{s_1=0}^{N} g(s_1) e^{-\beta s_1 E_1} e^{-\beta (N - s_1) E_2} \tag{2.6}$$

where g is the degeneracy of states for each value of s_1. The number of ways that we can have s_1 systems in the first spin state is the number of ways that we can choose s_1 systems out of the N total systems, so it is equal to the binomial coefficient $\binom{N}{s_1}$. This tells us that:

$$Z = \sum_{s_1=0}^{N} \binom{N}{s_1} e^{-\beta s_1 E_1} e^{-\beta (N - s_1) E_2} \tag{2.7}$$

The binomial theorem tells us that the right hand side of this expression is the series expansion of a binomial in $e^{-\beta E_1}$ and $e^{-\beta E_2}$. The series subsequently evaluates to the following:

$$\boxed{Z = \left(e^{-\beta E_1} + e^{-\beta E_2}\right)^N} \tag{2.8}$$

Note: *This is a very important result. The preceding argument can be generalized to tell us that the partition function of a system made up of non-interacting subsystems is equal to the product of the partition function of the subsystems.*

The average energy U of the system is given by:

$$U = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} \tag{2.9}$$

Substituting in our expression for the partition function:

$$U = -\frac{1}{\left(e^{-\beta E_1} + e^{-\beta E_2}\right)^N} \frac{\partial}{\partial \beta} \left(e^{-\beta E_1} + e^{-\beta E_2}\right)^N \tag{2.10}$$

Next, we evaluate the derivative and simplify, which yields:

$$\boxed{U = \frac{N(E_1 e^{-E_1/k_BT} + E_2 e^{-E_2/k_BT})}{e^{-E_1/k_BT} + e^{-E_2/k_BT}}} \tag{2.11}$$

Does this make sense? *This is the result we would naively expect, as it is just the probability weighted average of the energies.*

Next, we shall calculate the specific heat at constant volume C_V. By definition, we know that:

$$C_V(T) = \left(\frac{\partial U}{\partial T}\right)_V \tag{2.12}$$

In terms of β, we can write this as:

$$C_V(T) = -\frac{1}{k_BT^2}\left(\frac{\partial U}{\partial \beta}\right)_V \tag{2.13}$$

Substituting in our expression for the internal energy, we find:

$$C_V(T) = -\frac{N}{k_BT^2}\frac{\partial}{\partial \beta}\frac{E_1 e^{-\beta E_1} + E_2 e^{-\beta E_2}}{e^{-\beta E_1} + e^{-\beta E_2}} \tag{2.14}$$

We can now use the quotient rule to evaluate the derivative:

$$C_V(T) = \frac{N\left[(E_1 e^{-\beta E_1} + E_2 e^{-\beta E_2})^2 - (E_1^2 e^{-\beta E_1} + E_2^2 e^{-\beta E_2})(e^{-\beta E_1} + e^{-\beta E_2})\right]}{k_BT^2(e^{-\beta E_1} + e^{-\beta E_2})^2} \tag{2.15}$$

We can now simplify this expression to obtain the following:

$$\boxed{C_V(T) = \frac{N(E_1 - E_2)^2}{k_BT^2(e^{(E_1-E_2)/2k_BT} + e^{(E_2-E_1)/2k_BT})^2}} \tag{2.16}$$

(b) *Find the expressions for U and C_V in the limit where T is very large. That is, find the limit as T goes to infinity along with the first nonvanishing correction for very large but finite T.*

To find the limiting behavior of the expressions obtained in the previous part, it is useful to write them as hyperbolic trigonometric functions. We can rewrite our expression for U as:

$$U = \frac{N(E_1 e^{\beta(E_2-E_1)/2} + E_2 e^{\beta(E_1-E_2)/2})}{e^{-\beta(E_1-E_2)/2} + e^{-\beta(E_2-E_1)/2}} \tag{2.17}$$

Comparing the denominator to the definition of the hyperbolic cosine, we see that:

$$\begin{aligned} U = \frac{N}{2}\operatorname{sech}\left(\frac{\beta(E_1 - E_2)}{2}\right)\Bigg(&\frac{E_1 + E_2}{2}e^{-\beta(E_1-E_2)/2} + \frac{E_1 - E_2}{2}e^{-\beta(E_1-E_2)/2} \\ &+ \frac{E_1 + E_2}{2}e^{\beta(E_1-E_2)/2} + \frac{E_2 - E_1}{2}e^{\beta(E_1-E_2)/2}\Bigg) \end{aligned} \tag{2.18}$$

Next, we can write the quantities inside the parentheses in terms of hyperbolic trigonometric functions, which yields:

$$U = \frac{N(E_1 + E_2)}{2} + \frac{N(E_1 - E_2)}{2} \tanh\left(\frac{\beta(E_2 - E_1)}{2}\right) \tag{2.19}$$

We now expand this expression about $\beta = 0$, which, to leading order, gives us the equation:

$$U \approx \frac{N(E_1 + E_2)}{2} - \frac{N(E_1 - E_2)^2}{4} \operatorname{sech}^2(0)\beta \tag{2.20}$$

Thus, we conclude that in the large temperature regime, the average energy is given by:

$$\boxed{U(T) \approx \frac{N(E_1 + E_2)}{2} - \frac{N(E_1 - E_2)^2}{4k_B T}} \tag{2.21}$$

Does this make sense? *In the inifinite temperature limit, there is an equal probability of being in either of the two states. Thus, the leading order term must be the average of the two energies multiplied by the number of particles. In the finite temperature regime, there is a higher probability of particles being in a lower energy state, with a weight that only depends on the energy difference between the two states, which results in a negative correction to the leading order term. Notice further that this correction will be the same regardless of whether E_1 or E_2 has a greater energy, so it must be invariant under their exchange. This, combined with the fact that the term must have dimensions of energy means that it must be quadratic in the energy difference.*

We shall now perform a similar procedure with the specific heat. Rewriting our expression from the previous part in terms of hyperbolic trigonometric functions, we have:

$$C_V(T) = \frac{Nk_B(E_1 - E_2)^2\beta^2}{4} \operatorname{sech}^2\left(\frac{\beta(E_1 - E_2)}{2}\right) \tag{2.22}$$

Firt, we recognize that as β goes to zero, this expression vanishes. Notice that the only dependence on β that is not in the form of a polynomial comes from the hyperbolic secant. Then, we only need to expand that term about $\beta = 0$. Notice that:

$$\operatorname{sech} x = \frac{2}{e^{-x} + e^{x}} \approx \frac{2}{2 + 2x + \mathcal{O}(x^2)} = 1 + \mathcal{O}(x) \tag{2.23}$$

Then, to lowest non-trivial order, we have:

$$\boxed{C_V(T) \approx \frac{Nk_B(E_1 - E_2)^2}{4k_B T^2}} \tag{2.24}$$

Does this make sense? *In the inifinite temperature limit, an increase in temperature does not affect the state populations, and subsequently does not alter the energy of the system. Thus, the leading order term must vanish. In the finite temperature regime an increase in temperature will change the probabilities of state population according to the difference in the energies. Using a similar argument as with the average energy, we see that the first non-trivial term must depend on the energies as some function of $(E_1 - E_2)^2$.*